Differential-difference equation for call option price in jump process model

Math 622

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Reading material: Shreve Section 11.7

1 Heuristic

Suppose $S(t)$ satisfies

$$dS(t) = \alpha S(t)dt + \sigma S(t)dM(t),$$

where $M(t) = N(t) - \lambda t$ is a compensated Poisson process under $\mathbb{P}$.

From the change of measure section, we learned that under the risk neutral measure $\mathbb{Q}$, $S(t)$ has the dynamic:

$$dSt = (r - \tilde{\lambda}\sigma)S(t)dt + \sigma S(t-)dN(t),$$

where $\tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}$ and N is a Poisson process with rate $\tilde{\lambda}$ under $\mathbb{Q}$.

The call option price $V(t)$, where $V(T) = (S(T) - K)^+$ can be written as

$$\begin{aligned} V(t) &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}(t)\Big] \\ &= c(t, S(t)), \end{aligned}$$

where

$$c(t, x) := e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[(xe^{(r-\tilde{\lambda}\sigma)(T-t)+\log(1+\sigma)(N(T)-N(t))} - K)^+\Big].$$

As in the Black-Scholes model, we want to derive an equation that $c(t, x)$ satisfies. *The key principle* here (albeit being heurisitc) is to apply Ito's formula to $e^{-rt}c(t, S(t))$ to achieve

$$de^{-rt}c(t, S(t)) = f(t, c(t, S(t)))dt + \text{ something } dM(t),$$

where $M(t)$ *is a* $\mathbb{Q}$*-martingale.* Then the equation that we look for is

$$f(t, c(t, S(t)) = 0.$$

The heuristic reason is $e^{-rt}c(t, S(t))$ is a $\mathbb{Q}$-martingale by being a conditional expectation. Therefore, its drift has to be 0.

2 Model with Poisson random source

Suppose $S(t)$ satisfies

$$dS(t) = \alpha S(t)dt + \sigma S(t)dM(t),$$

where $M(t) = N(t) - \lambda t$ is a compensated Poisson process under $\mathbb{P}$.

Aplly Ito's formula to $e^{-rt}c(t, S(t))$, recognizing there is no Brownian motion component, we have

$$\begin{aligned}
e^{-rt}c(t, S(t)) &= \int_0^t -re^{-ru}c(u, S(u))du + e^{-ru}\frac{\partial}{\partial t}c(u, S(u))du + e^{-ru}\frac{\partial}{\partial x}c(u, S(u))dS^c(u) \\
&\quad + \sum_{0<u\leq t} e^{-ru}[c(u, S(u)) - c(u-, S(u-))] \\
&= \int_0^t e^{-ru}\Big[-rc(u, S(u)) + \frac{\partial}{\partial t}c(u, S(u)) + \frac{\partial}{\partial x}c(t, S(u))(r - \tilde{\lambda}\sigma)S(u)\Big]du \\
&\quad + \sum_{0<u\leq t} e^{-ru}[c(u, S(u)) - c(u, S(u-))].
\end{aligned}$$

We need to rewrite $\sum_{0<u\leq t} e^{-ru}[c(u, S(u)) - c(u, S(u-))]$ as it is *not in differential form.* Two key observations will help us here:

(i) $S(u) = (1 + \sigma\Delta N(u))S(u-) = (1 + \sigma)S(u-)$.

(ii) $c(u, S(u))$ jumps at the same points as $S(u)$, which in turn jumps at the same points as $N(u)$. Again keep in mind that $\Delta N(u) = 1$.

Thus

$$\begin{aligned}
&\sum_{0<u\leq t} e^{-ru}[c(u, S(u)) - c(u, S(u-))] = \sum_{0<u\leq t} e^{-ru}[c(u, S(u-)(1+\sigma)) - c(u, S(u-))] \\
= &\int_0^t e^{-ru}[c(u, S(u-)(1+\sigma)) - c(u, S(u-))]dN(u),
\end{aligned}$$

where the first equality uses observations (i) and second equality uses observation (ii).

Putting all these together gives

$$
\begin{aligned}
e^{-rt}c(t,S(t)) &= \int_0^t e^{-ru}\Big[-rc(u,S(u)) + \frac{\partial}{\partial t}c(u,S(u)) + \frac{\partial}{\partial x}c(t,S(u))(r-\tilde{\lambda}\sigma)S(u)\Big]du \\
&+ \int_0^t e^{-ru}[c(u,S(u-)(1+\sigma)) - c(u,S(u-))]dN(u).
\end{aligned}
$$

The last thing to do is to change $dN(u)$ to $dM(u)$ for some martingale M. This is easy: we only need to subtract and add $\tilde{\lambda}du$ to $dN(u)$. So finally

$$
\begin{aligned}
e^{-rt}c(t,S(t)) &= \int_0^t e^{-ru}\Big[-rc(u,S(u)) + \frac{\partial}{\partial t}c(u,S(u)) + \frac{\partial}{\partial x}c(t,S(u))(r-\tilde{\lambda}\sigma)S(u) \\
&+[c(u,S(u-)(1+\sigma)) - c(u,S(u-))]\tilde{\lambda}\Big]du \\
&+ \int_0^t e^{-ru}[c(u,S(u-)(1+\sigma)) - c(u,S(u-))]dM(u) \\
&= \int_0^t e^{-ru}\Big[-rc(u,S(u)) + \frac{\partial}{\partial t}c(u,S(u)) + \frac{\partial}{\partial x}c(t,S(u))(r-\tilde{\lambda}\sigma)S(u) \\
&+[c(u,S(u)(1+\sigma)) - c(u,S(u))]\tilde{\lambda}\Big]du \\
&+ \int_0^t e^{-ru}[c(u,S(u-)(1+\sigma)) - c(u-,S(u-))]dM(u),
\end{aligned}
$$

where in the second equality we uses the fact that we are integrating with respect to du so using $S(u-)$ or $S(u)$ gives the same result.

Now apply the principle in Section (1) we get

Theorem 2.1. *The call option price $c(t,x)$ in the model of this section satisfies the differential difference equation*

$$
\begin{aligned}
-rc(t,x) &+ \frac{\partial}{\partial t}c(t,x) + (r-\tilde{\lambda}\sigma)x\frac{\partial}{\partial x}c(t,x) \\
&+ \tilde{\lambda}[c(t,x(1+\sigma)) - c(t,x)] = 0, 0 \le t < T, x > 0 \\
c(T,x) &= (x-K)^+, x > 0.
\end{aligned}
$$

3 Model with compound Poisson random source

Suppose $S(t)$ has the dynamic:

$$dSt = (r - \tilde{m}\sigma)S(t)dt + \sigma S(t-)dQ(t),$$

where $Q(t)$ is a compound Poisson process with rate $\mathbb{E}^{\mathbb{Q}}(Q(1)) = 1$. We also assume that $Q(t) = \sum_{i=1}^{N(t)} Y_i$ where each Y_i takes discrete distribution with values $y_1, y_2, ..., y_m$.

Following the same procedure as the above section, apply Ito's formula to $e^{-rt}c(t, S(t))$ gives

$$
\begin{aligned}
e^{-rt}c(t,S(t)) &= \int_0^t -re^{-ru}c(u,S(u))du + e^{-ru}\frac{\partial}{\partial t}c(u,S(u))du + e^{-ru}\frac{\partial}{\partial x}c(u,S(u))dS^c(u) \\
&\quad + \sum_{0<u\le t} e^{-ru}[c(u,S(u)) - c(u-,S(u-))] \\
&= \int_0^t e^{-ru}\Big[-rc(u,S(u)) + \frac{\partial}{\partial t}c(u,S(u)) + \frac{\partial}{\partial x}c(t,S(u))(r-\tilde{m}\sigma)S(u)\Big]du \\
&\quad + \sum_{0<u\le t} e^{-ru}[c(u,S(u)) - c(u,S(u-))].
\end{aligned}
$$

Now by the Poisson process decomposition, we can write

$$Q(t) = \sum_{i=1}^{m} y_i N_i(t),$$

where each $N_i(t)$ is a Poisson process with rate $\tilde{\lambda}_i, i = 1, ..., m$ under $\mathbb{Q}$. An important fact here is that since N_i's are independent, *they do not jump at the same time*. So at all jump point of Q:

$$1 + \sigma\Delta Q(t) = 1 + \sigma y_i \Delta N_i(t), \text{ for some } i.$$

Thus we have,

$$
\begin{aligned}
&\sum_{0<u\le t} e^{-ru}[c(u,S(u)) - c(u,S(u-))] = \sum_{0<u\le N(t)} e^{-ru}[c(u,S(u-)(1+\sigma\Delta Q(u))) - c(u,S(u-))] \\
&= \sum_{i=1}^{m}\Big[\sum_{0<u\le t} e^{-ru}[c(u,S(u-)(1+\sigma y_i)) - c(u,S(u-))]\Delta N_i(u)\Big] \\
&= \sum_{i=1}^{m}\Big[\int_0^t e^{-ru}[c(u,S(u-)(1+\sigma y_i)) - c(u,S(u-))]dN_i(u)\Big].
\end{aligned}
$$

So

$$
\begin{aligned}
e^{-rt}c(t,S(t)) &= \int_0^t e^{-ru}\Big[-rc(u,S(u)) + \frac{\partial}{\partial t}c(u,S(u)) + \frac{\partial}{\partial x}c(t,S(u))(r-\tilde{m}\sigma)S(u) \\
&\quad + \sum_{i=1}^{m}[c(u,S(u)(1+\sigma y_i)) - c(u,S(u))]\tilde{\lambda}_i\Big]du \\
&\quad + \int_0^t e^{-ru}[c(u,S(u)) - c(u,S(u-))]dM(u),
\end{aligned}
$$

where

$$M(t) = \sum_{i=1}^{m} N_i(t) - \tilde{\lambda}_i t$$

is a $\mathbb{Q}$-martingale.

Setting the dt part to be 0 gives the following:

Theorem 3.1. *The call option price $c(t,x)$ in the model of this section satisfies the differential difference equation*

$$\begin{aligned} -rc(t,x) &+ \frac{\partial}{\partial t}c(t,x) + (r - \tilde{m}\sigma)x\frac{\partial}{\partial x}c(t,x) \\ &+ \sum_{i=1}^{m}[c(t, x(1+\sigma y_i)) - c(t,x)]\tilde{\lambda}_i = 0, 0 \le t < T, x > 0 \\ c(T,x) &= (x-K)^+, x > 0. \end{aligned}$$

4 Model with compound Poisson and Brownian motion random source

Suppose $S(t)$ has the dynamic:

$$dSt = (r - \tilde{m})S(t)dt + S(t-)dQ(t) + \sigma S(t)d\tilde{W}(t),$$

where $Q(t)$ is a compound Poisson process with rate $\mathbb{E}^{\mathbb{Q}}(Q(1)) = 1$ and $\tilde{W}(t)$ is a $\mathbb{Q}$ Brownian motion. We also assume that $Q(t) = \sum_{i=1}^{N(t)} Y_i$ where each Y_i takes discrete distribution with values $y_1, y_2, ..., y_m$.

Following the same procedure as the above section, apply Ito's formula to $e^{-rt}c(t, S(t))$ gives

$$\begin{aligned} e^{-rt}c(t,S(t)) &= \int_0^t -re^{-ru}c(u,S(u))du + e^{-ru}\frac{\partial}{\partial t}c(u,S(u))du + e^{-ru}\frac{\partial}{\partial x}c(u,S(u))dS^c(u) \\ &+\frac{1}{2}e^{-ru}\frac{\partial^2}{\partial x^2}c(u,S(u))\sigma^2 S^2(u)du + \sum_{0<u\le t} e^{-ru}[c(u,S(u)) - c(u-,S(u-))] \\ &= \int_0^t e^{-ru}\Big[-rc(u,S(u)) + \frac{\partial}{\partial t}c(u,S(u)) + \frac{\partial}{\partial x}c(t,S(u))(r-\tilde{m})S(u) \\ &+\frac{1}{2}\frac{\partial^2}{\partial x^2}c(u,S(u))\sigma^2 S^2(u)\Big]du \\ &+\int_0^t e^{-ru}\frac{\partial}{\partial x}c(t,S(u))S(u)d\tilde{W}(u) + \sum_{0<u\le t} e^{-ru}[c(u,S(u)) - c(u,S(u-))]. \end{aligned}$$

Follow the same exact analysis for $\sum_{0<u\le t} e^{-ru}[c(u,S(u)) - c(u,S(u-))]$ as in section (3) we have

$$\begin{aligned} e^{-rt}c(t,S(t)) &= \int_0^t e^{-ru}\Big[-rc(u,S(u)) + \frac{\partial}{\partial t}c(u,S(u)) + \frac{\partial}{\partial x}c(t,S(u))(r-\tilde{m})S(u) \\ &+\frac{1}{2}\frac{\partial^2}{\partial x^2}c(u,S(u))\sigma^2 S^2(u) + \sum_{i=1}^{m}[c(u,S(u)(1+y_i)) - c(u,S(u))]\tilde{\lambda}_i\Big]du \\ &+\int_0^t e^{-ru}\frac{\partial}{\partial x}c(t,S(u))S(u)d\tilde{W}(u) + \int_0^t e^{-ru}[c(u,S(u)) - c(u,S(u-))]dM(u), \end{aligned}$$

where

$$M(t) = \sum_{i=1}^{m} N_i(t) - \tilde{\lambda}_i t$$

is a $\mathbb{Q}$-martingale.

Setting the dt part to be 0 gives the following:

Theorem 4.1. *The call option price $c(t,x)$ in the model of this section satisfies the differential difference equation*

$$\begin{aligned} -rc(t,x) &+ \frac{\partial}{\partial t}c(t,x) + (r-\tilde{m})x\frac{\partial}{\partial x}c(t,x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}c(t,x)\sigma^2x^2 \\ &+ \sum_{i=1}^{m}[c(t,x(1+y_i)) - c(t,x)]\tilde{\lambda}_i = 0, 0 \le t < t, x > 0; \\ c(T,x) &= (x-K)^+, x > 0. \end{aligned}$$